Problem from my notes.

Inequality for coefficients of one trigonometric polynomial.

https://www.linkedin.com/feed/update/urn:li:activity:6739795027867922432

Known that $\sum_{k=1}^{n} a_k \cos(kx) \geq -1$ for all $x \in \mathbb{R}$. Prove that $\sum_{k=1}^{n} a_k \leq n$.

Solution by Arkady Alt, San Jose, California, USA.

Let $\varepsilon := \cos\frac{2\pi}{n+1} + i\sin\frac{2\pi}{n+1}$. Then $1 + \varepsilon^r + \varepsilon^{2r} + \ldots + \varepsilon^{nr} = \frac{(\varepsilon^r)^{n+1} - 1}{\varepsilon^r - 1} = \frac{(\varepsilon^{n+1})^r - 1}{\varepsilon^r - 1} = \frac{1-1}{\varepsilon^r - 1} = 0$ for any $r = 1, 2, \ldots, n$.

Thus, $\sum_{k=1}^{n} \varepsilon^{kr} = -1 \iff \sum_{k=1}^{n}\left(\cos\frac{2kr\pi}{n+1} + i\sin\frac{2kr\pi}{n+1}\right) = -1$ implies

$$\sum_{k=1}^{n} \cos\frac{2kr\pi}{n+1} = -1 \text{ for each } r = 1, 2, \ldots, n.$$

Since $\sum_{k=1}^{n} a_k \cos(nx) \geq -1$ for any $x \in \mathbb{R}$ then in particular for $x = \frac{2r\pi}{n+1}$ we have inequalities $\sum_{k=1}^{n} a_k \cos\left(\frac{2rk\pi}{n+1}\right) \geq -1, r = 1, 2, \ldots, n.$

Therefore, $\sum_{r=1}^{n}\sum_{k=1}^{n} a_k \cos\left(\frac{2rk\pi}{n+1}\right) \geq -n$ and since $\sum_{r=1}^{n}\sum_{k=1}^{n} a_k \cos\left(\frac{2rk\pi}{n+1}\right) = \sum_{k=1}^{n}\sum_{r=1}^{n} a_k \cos\left(\frac{2rk\pi}{n+1}\right) = \sum_{k=1}^{n} a_k \sum_{r=1}^{n} \cos\left(\frac{2rk\pi}{n+1}\right) = \sum_{k=1}^{n} a_k(-1) = -\sum_{k=1}^{n} a_k$

then $-\sum_{k=1}^{n} a_k \geq -n \iff \sum_{k=1}^{n} a_k \leq n$.

Remark.

Before the general case first was considered two particular cases $n = 2, 3$.

1. Let $n = 2$. Then we have $a_1 \cos x + a_2 \cos 2x \geq -1, x \in \mathbb{R}$ and in particular for $x = \frac{2\pi}{3}$ we obtain $a_1 \cos\frac{2\pi}{3} + a_2 \cos\frac{4\pi}{3} \geq -1 \iff a_1\left(-\frac{1}{2}\right) + a_2\left(-\frac{1}{2}\right) \geq -1 \iff a_1 + a_2 \leq 2$.
2. Let $n = 3$. Then we have $a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \geq -1, x \in \mathbb{R}$.

In particular for:

$x = \frac{\pi}{2}$ we obtain $a_1 \cos\frac{\pi}{2} + a_2 \cos\frac{2\pi}{2} + a_3 \cos\frac{3\pi}{2} \geq -1 \iff -a_2 \geq -1 \iff a_2 \leq 1$;

$x = \pi$ we obtain $a_1 \cos\pi + a_2 \cos 2\pi + a_3 \cos 3\pi \geq -1 \iff -a_1 + a_2 - a_3 \geq -1$.

then $-a_1 - a_2 - a_3 = (-a_1 + a_2 - a_3) - 2a_2 \geq -1 + (-2) = -3 \iff a_1 + a_2 + a_3 \leq 3$.